Fast Fourier Transforms and Signal Processing

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Introduction

- I'm going to assume here that you know what an FFT is and what you might use it for.
- So my intent is to show you how to implement FFTs in Matlab
- In practice, it is trivial to calculate an FFT in Matlab, but takes a bit of practice to use it appropriately
- This is the same in every tool I've ever used

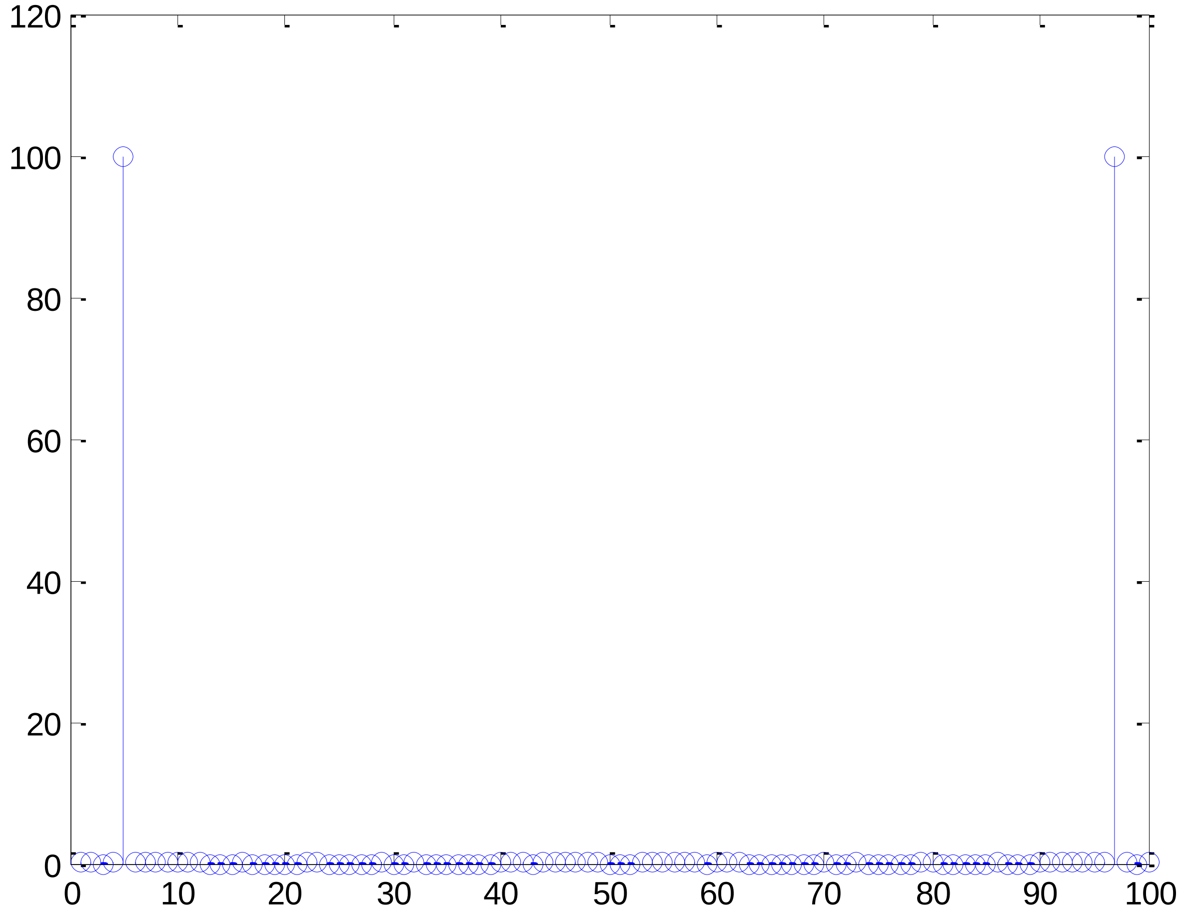
FFTs of Functions

- We can sample a function and then take the FFT to see the function in the frequency domain
- Of course, we must sample often enough to avoid losing content
- The script on the following page samples a sine wave

Sampling a sine wave

```
fo = 4; %frequency of the sine wave
Fs = 100; %sampling rate
Ts = 1/Fs; %sampling time interval
t = 0:Ts:1-Ts;
n = length(t); %number of samples
y = 2*sin(2*pi*fo*t);
plot(t,y)
YfreqDomain = fft(y);
stem(abs(YfreqDomain));
axis([0,100,0,120])
```

Output



Correlating x-axis with frequencies

- The previous plot just uses the element number as the row axis.
- In reality, each data point represents a frequency.
- These frequencies are calculated from the sampling rate
- The routine on the next page puts this together.
  - Send a dataset and sampling rate

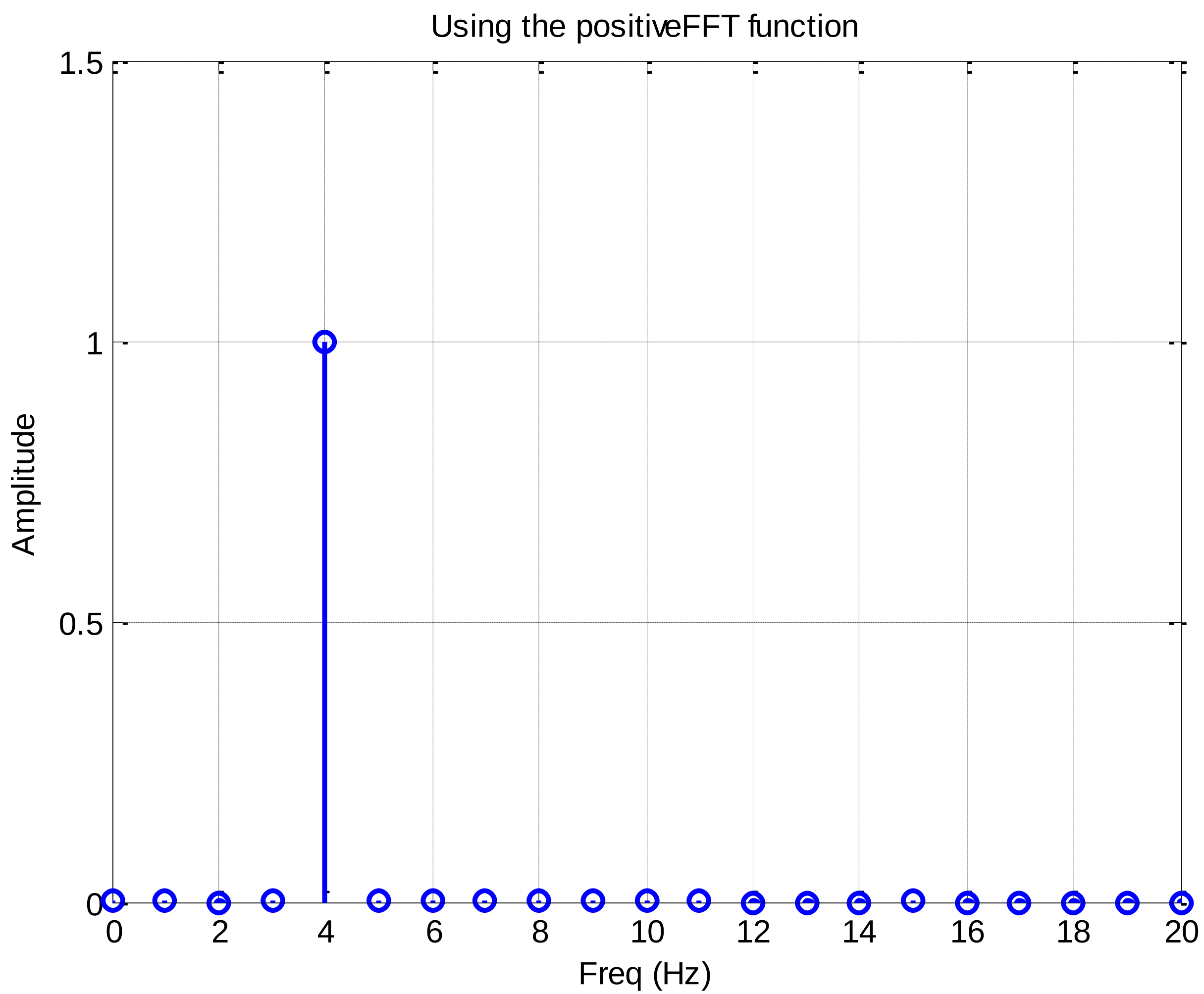
A Useful Function

```
function [X,freq]=positiveFFT(x,Fs)
N=length(x);
k=0:N-1;
T=N/Fs;
freq=k/T; %create the frequency range
X=fft(x)/N; % normalize the data
cutOff = ceil(N/2);
X = X(1:cutOff);
freq = freq(1:cutOff);
```

Key Calling Statements

```
fo = 4; %frequency of the sine wave
Fs = 100; %sampling rate
Ts = 1/Fs; %sampling time interval
t = 0:Ts:1-Ts;
n = length(t); %number of samples
y = 2*sin(2*pi*fo*t);
[YfreqD,freqRng] = positiveFFT(y,Fs);
stem(freqRng,abs(YfreqD));
```

New Plot



Using the positiveFFT function
1.5
1
0.5
0
Amplitude
0
2
4
6
8
10
12
14
16
18
20
Freq (Hz)

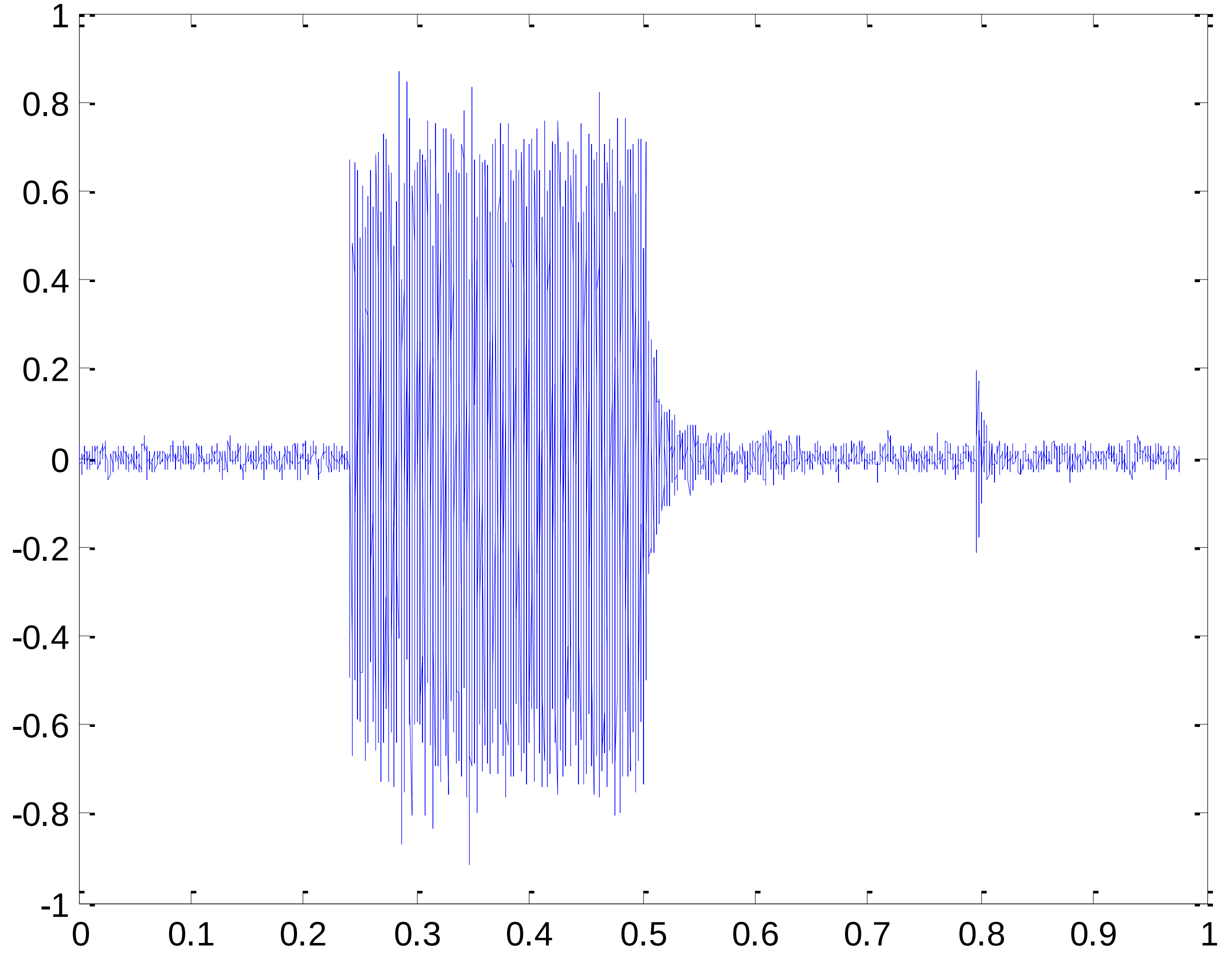
FFT of Imported Data

- We can read in sampled data and a sample rate and then take an FFT
- The file **touchtone.mat** contains a ringtone waveform for an 11 digit phone number (from Moler text)
- The commands to create a vector appropriate for sampling are on the next slide

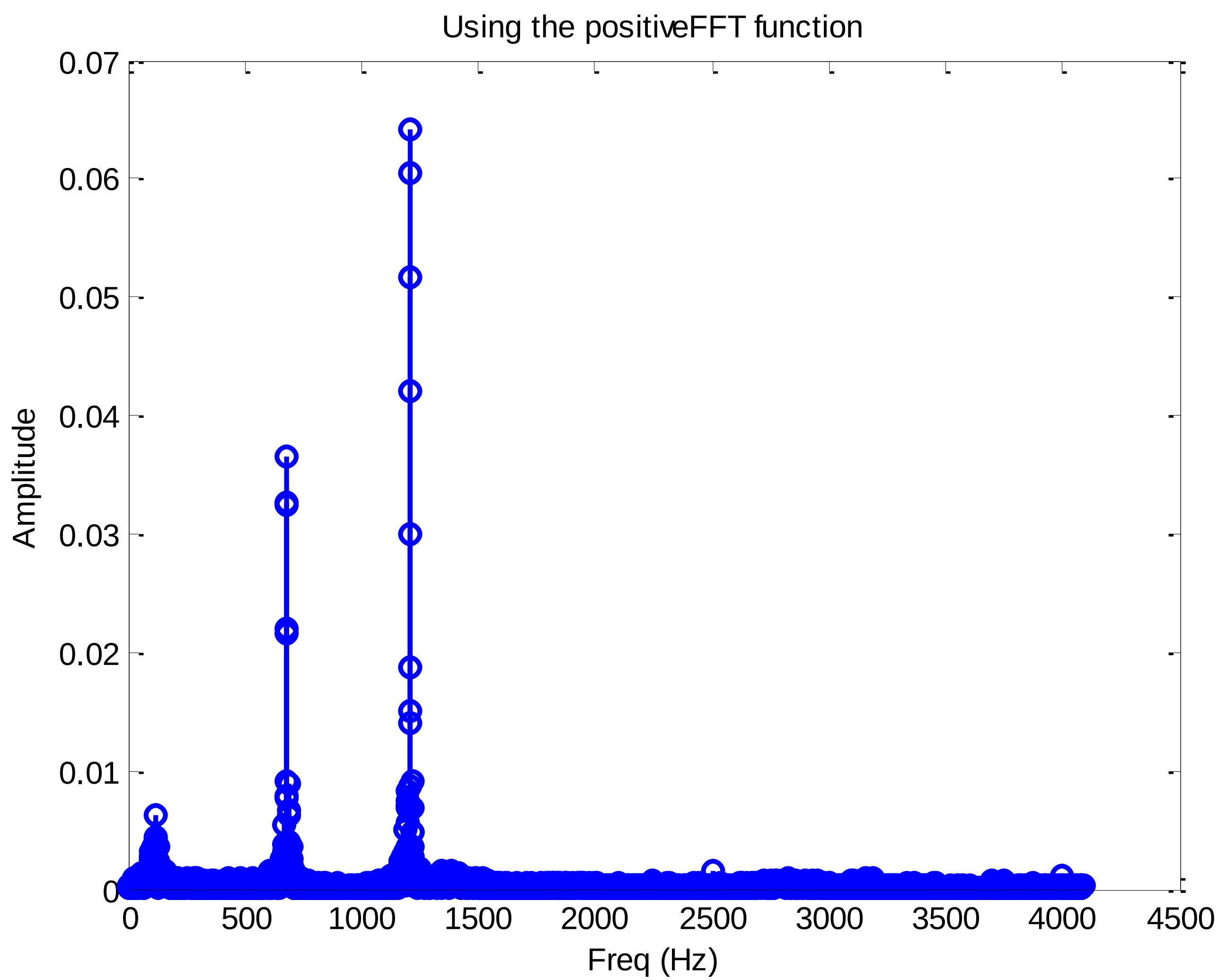
Script for first number dialed

```
load touchtone
Fs=y.fs
n = length(y.sig);      % number of samples
t = (0:n-1)/y.fs;   % Time for entire signal
y = double(y.sig)/128;
t=t(1:8000) % take first 8,000 samples
y=y(1:8000)
plot(t,y)
```

Time Signal

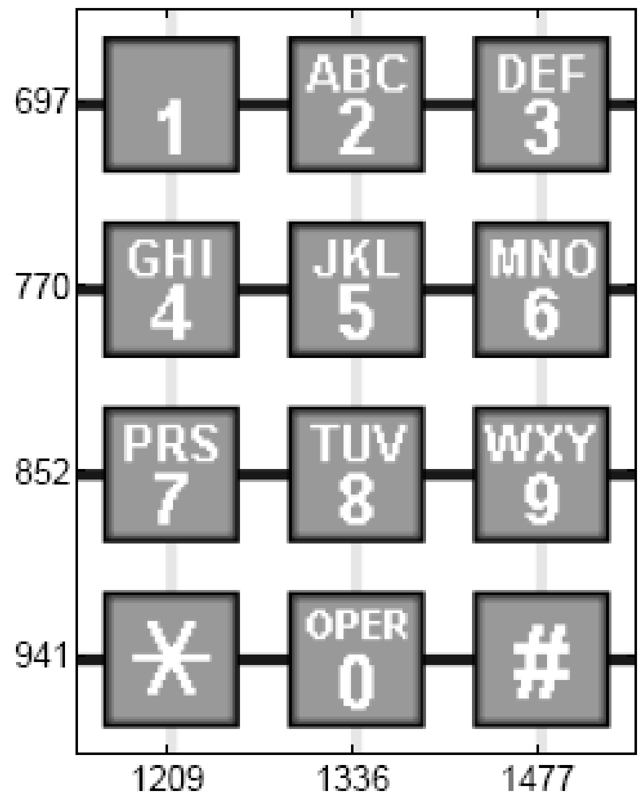


Output Spectrum



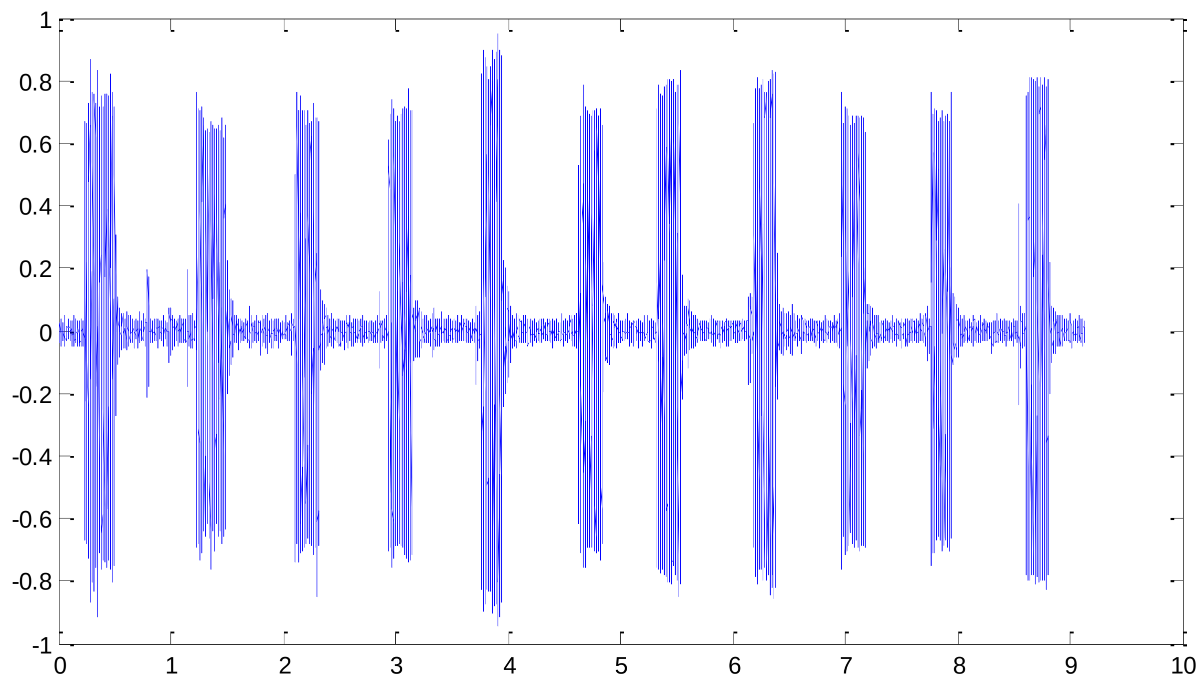
What number was dialed?

- To figure out which number was dialed, look at this grid



What is second number?

- Take the next set of data and figure out which number was dialed.
- Try points from 8,000 to 15,000



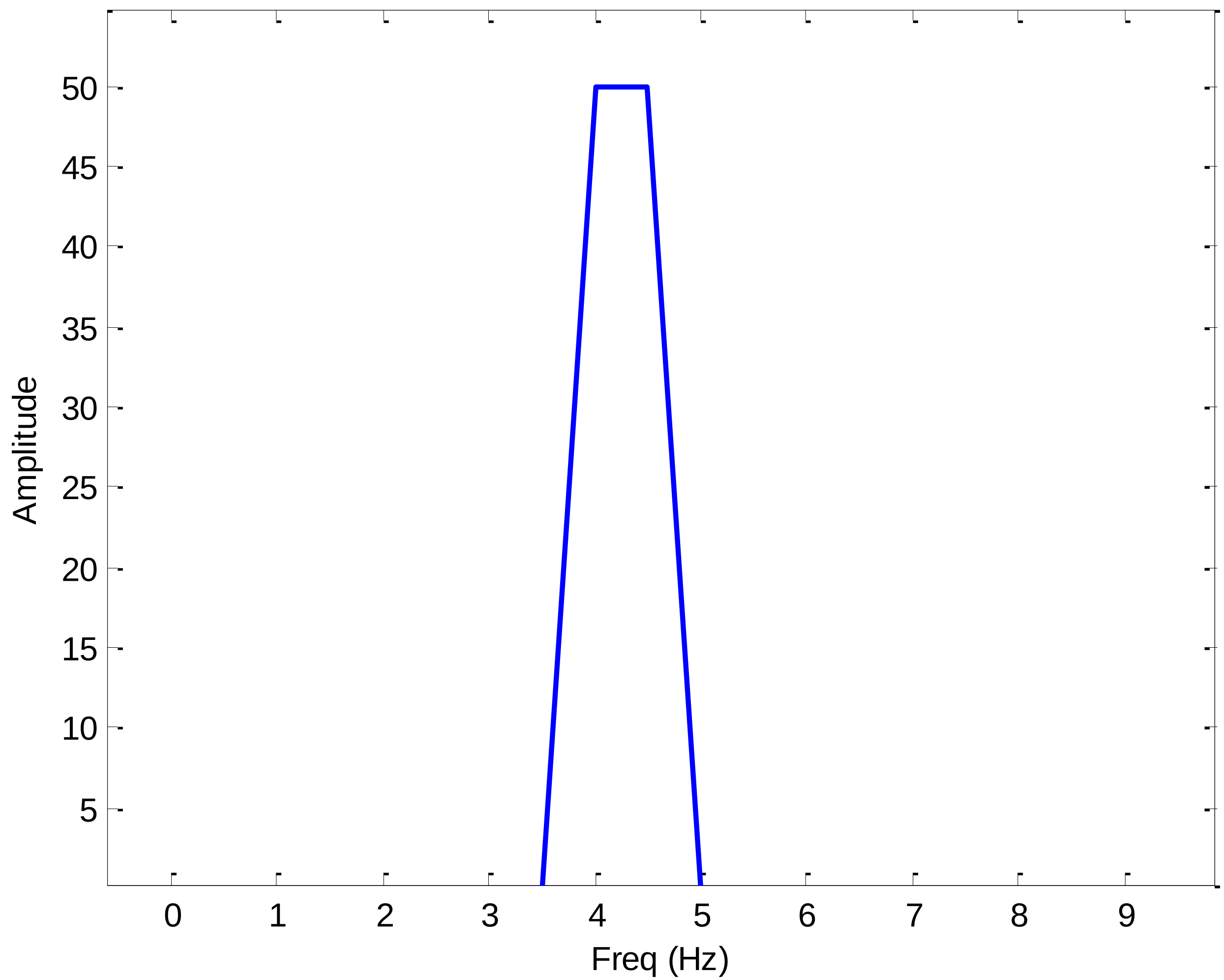
Zero Padding (blinkdagger.com)

- FFTs work with vectors containing a number of elements which is an even power of 2
- If you have data which is not a power of 2, you can fill with 0's
- This will get you faster performance and better resolution

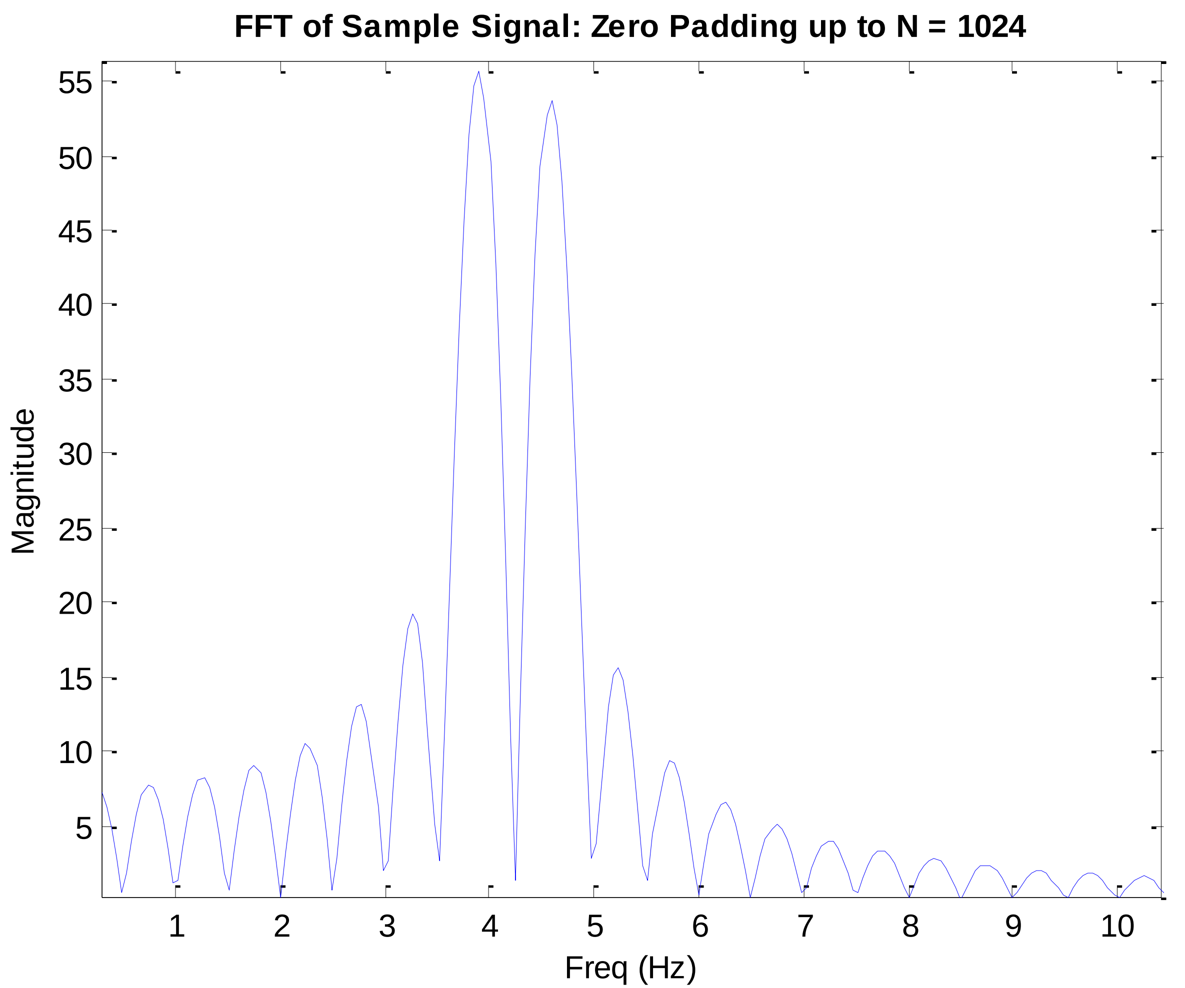
Example

- Beats: $y=\sin(2\pi f_1 t)+\sin(2\pi f_2 t)$
- Let f_1=4Hz and f_2=4.5Hz
- Sample at 100 Hz
- Take FFT with and without padding

Not Padded



Zero-Padded



FFT of Sample Signal: Zero Padding up to N = 1024
Magnitude
55
50
45
40
35
30
25
20
15
10
5
1
2
3
4
5
6
7
8
9
10
Freq (Hz)

Script

```
zeroPadFac= nextpow2(length(y)) + 3;
[a,b] = posFFTzeropad(y,Fs,2^zeroPadFac);
%
function [X,freq]=posFFTzeropad(x,Fs,N)
k=0:N-1;
T=N/Fs;
freq=k/T;
X=fft(x,N)/length(x);
cutOff = ceil(N/2);
X = X(1:cutOff);
freq = freq(1:cutOff);
```

Convolution

- Once we can do FFTs, we can do convolution
- Matlab has several built-in functions for this
- To convolve 2 vectors, it is just:
  w = conv(u,v)

The Convolution Algorithm

xtrans = fft([x zeros(1,length(y)-1)])

ytrans = fft([y zeros(1,length(x)-1)])

conv(x,y) = ifft(xtrans.*ytrans)

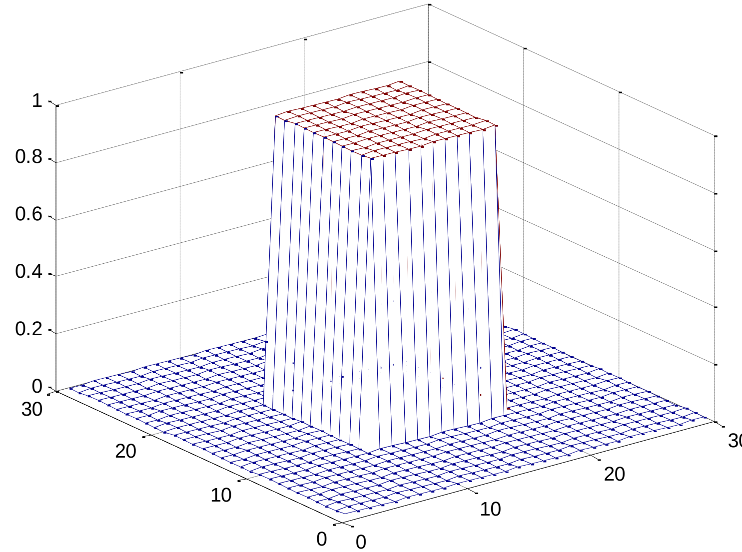
2-D Convolution

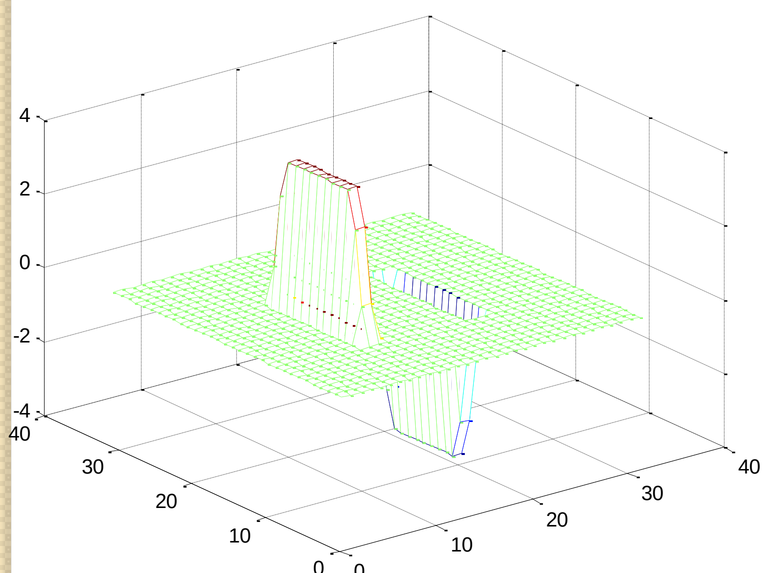
```
A = rand(3);
B = rand(4);
C = conv2(A,B)
```

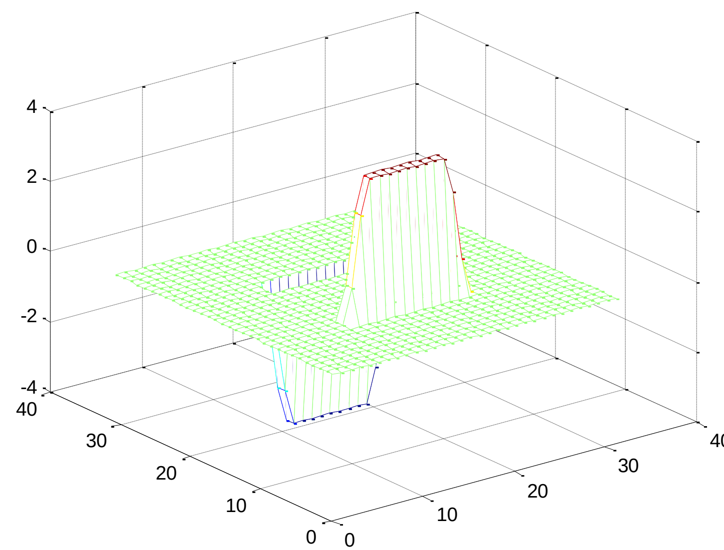
Example – edge-finding

```
s = [1 2 1; 0 0 0; -1 -2 -1];
A = zeros(30);
A(10:20,10:20) = ones(11);
mesh(A)
H = conv2(A,s);
figure
mesh(H)
V = conv2(A,s');
figure
mesh(V)
```

Results


1
0.8
0.6
0.4
0.2
0
30
20
10
0
0
10
20
30


4
2
0
-2
-4
40
30
20
10
0
0
10
20
30
40


4
2
0
-2
-4
40
30
20
10
0
0
10
20
30
40

Digital Filters

- Matlab has several filters built in
- One is the **filtfilt** command

What is filtfilt?

- This is a zero-phase, forward and reverse digital filter
- **y=filtfilt(b, a, x)**
- b and a define filter; x is the data to be filtered
- The length of x must be at least 3 times the order of the filter (max of length(a) or length(b) minus 1)

filtfilt algorithm

- The filtfilt algorithm is based on a difference equation
- Providing vectors a and b, determine the outcome of the filter
- The difference equation is:
- $y(n) = b(1)*x(n) + b(2)*x(n-1) + \ldots + b(nb+1)*x(n-nb) - a(2)*y(n-1) - \ldots - a(na+1)*y(n-na)$
- b operates on the input vector (x) and a operates on the output vector (y)

Butterworth Filters

- Matlab has tools to prepare these vectors defining digital filters
- One example is the Butterworth filter
- **[B,A] = butter (N,Wn,'high')** designs a highpass filter.
- N is order of filter
- Wn is normalized cutoff frequency
- B and A are sent to the filtfilt command to actually filter data

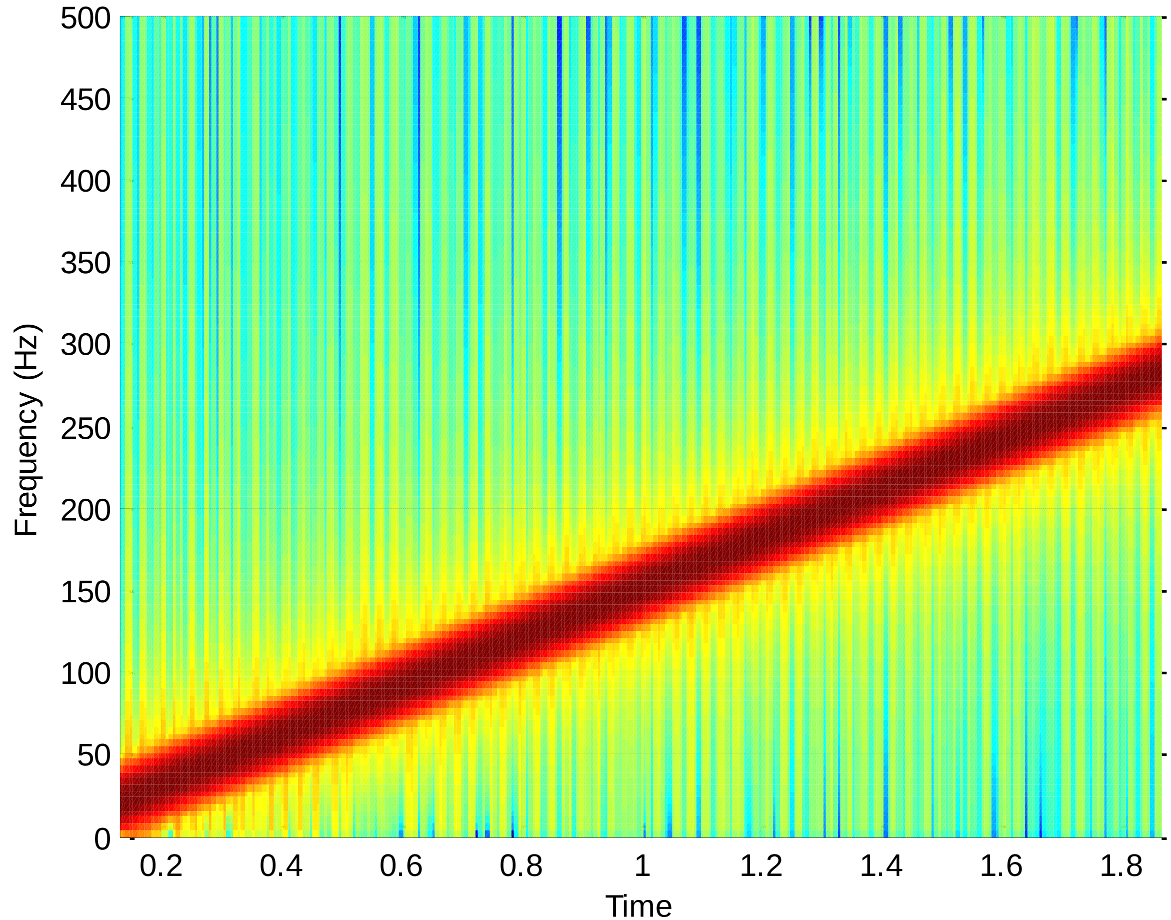
Butterworth Filters (cont.)

- **[B,A] = butter (N,Wn,'low')** designs a lowpass filter.
- **[B,A] = butter(N,Wn,'stop')** is a bandstop filter if Wn = [W1 W2].
- Note: cutoff frequency is frequency where magnitude of response is 1/sqrt(2)
- Hence, Wn is between 0 and 1, where 1 is the Nyquist frequency

Example

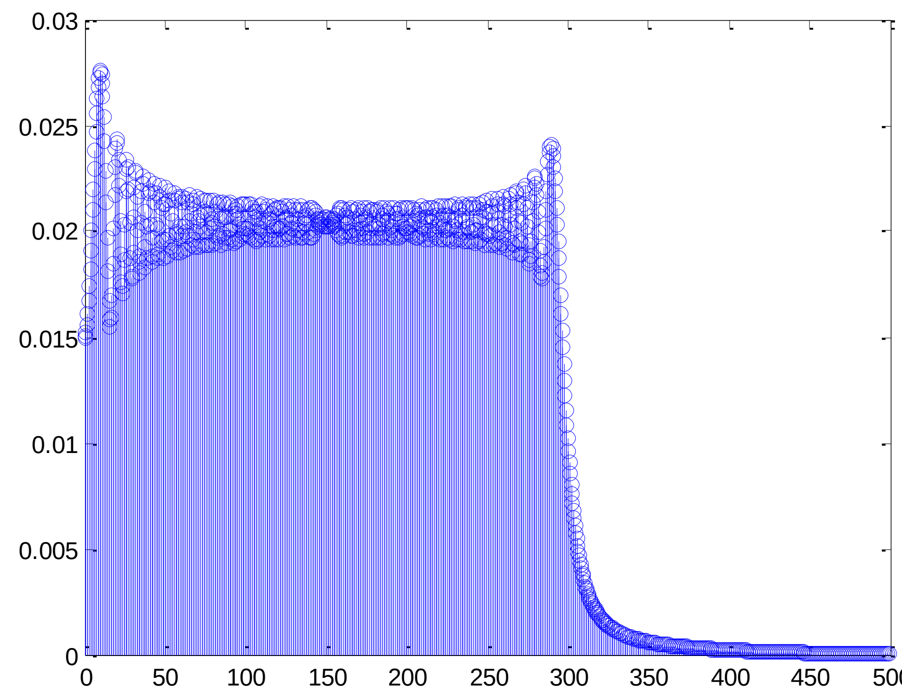
- Matlab has a built-in chirp signal
- **t=0:0.001:2**
- **y=chirp(t,0,1,150)**
- This samples a chirp for 2 seconds at 1 kHz – The frequency of the signal increases with time, starting at 0 and crossing 150 Hz at 1 second
- **sound(y)** will play the sound through your sound card
- **spectrogram(y,256,250,256,1E3,'yaxis')** will show time dependence of frequency
- Nyquist Frequency is f/2 or 500 Hz
- To set cutoff at 150 Hz, set Wn=150/500=0.3

Spectrogram



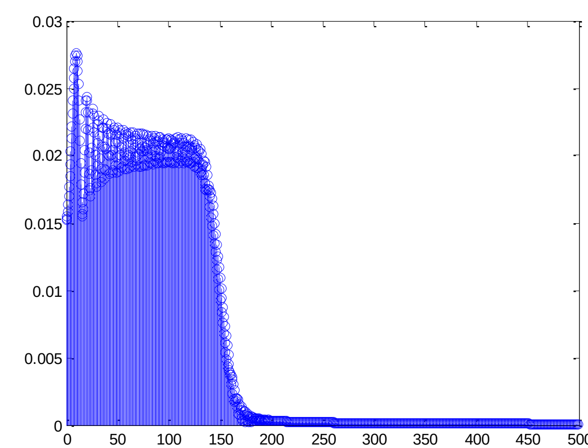
Example - continued

- Plot FFT of chirp
- **[YfreqD,freqRng] = positiveFFT(y,1000);**
- **stem(freqRng,abs(YfreqD));**



Example - continued

- Now use (lowpass) filter (10^{th} order Butterworth, cutoff at 150 Hz)
- **[b,a]=butter(10,0.3,'low')**
- **yfilt=filtfilt(b,a,y)**
- **[YfreqD,freqRng] = positiveFFT(yfilt,1000);**
- **stem(freqRng,abs(YfreqD));**



The script

```
Fs=1000;
t=0:1/Fs:2
y=chirp(t,0,1,150)
spectrogram(y,256,250,256,1E3,'yaxis')
[YfreqD,freqRng] = positiveFFT(y,Fs);
stem(freqRng,abs(YfreqD));
[b,a]=butter(10,0.3,'low');
yfilt=filtfilt(b,a,y);
[YfreqD,freqRng] =
  positiveFFT(yfilt,1000);
stem(freqRng,abs(YfreqD));
```

Practice

- Compare to a high pass filter with the same cutoff (150 Hz)
- Reminder: code for low pass filter is:
- **t=0:0.001:2**
- **y=chirp(t,0,1,150)**
- **[b,a]=butter(10,0.3,'low')**
- **yfilt=filtfilt(b,a,y)**
- **[YfreqD,freqRng] = positiveFFT(yfilt,1000);**
- **stem(freqRng,abs(YfreqD));**
- This is in fftscripts.m
- You'll need positiveFFT.m

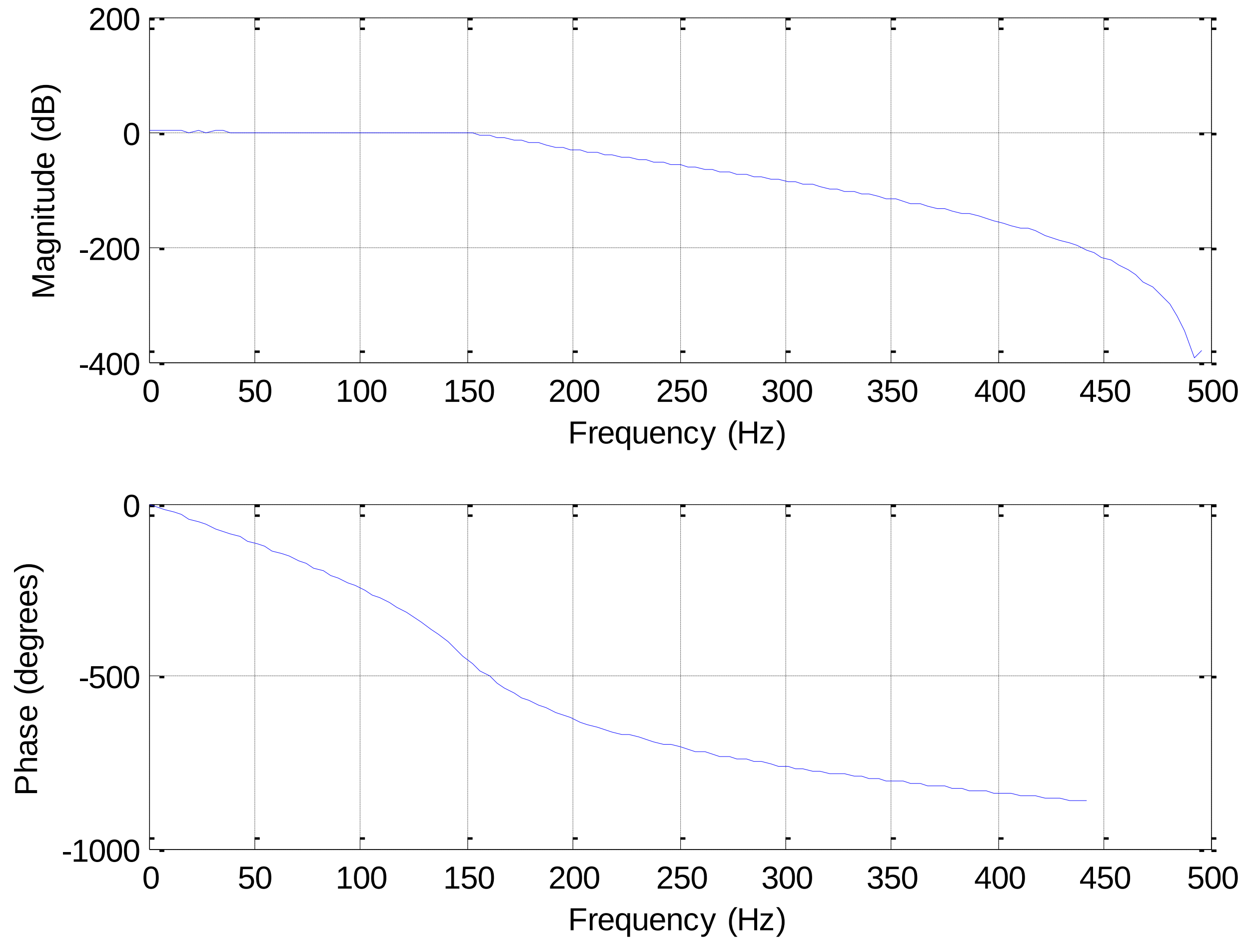
Filter Response

- To see a filter response, use the freqz or fvtool from the Signal Processing Toolkit
- From previous example: **freqz(b,a,128,Fs)** or **fvtool(b,a)**
- This will readily show you impulse response, step response, pole/zero plots, etc.

Do you have the SP Toolbox?

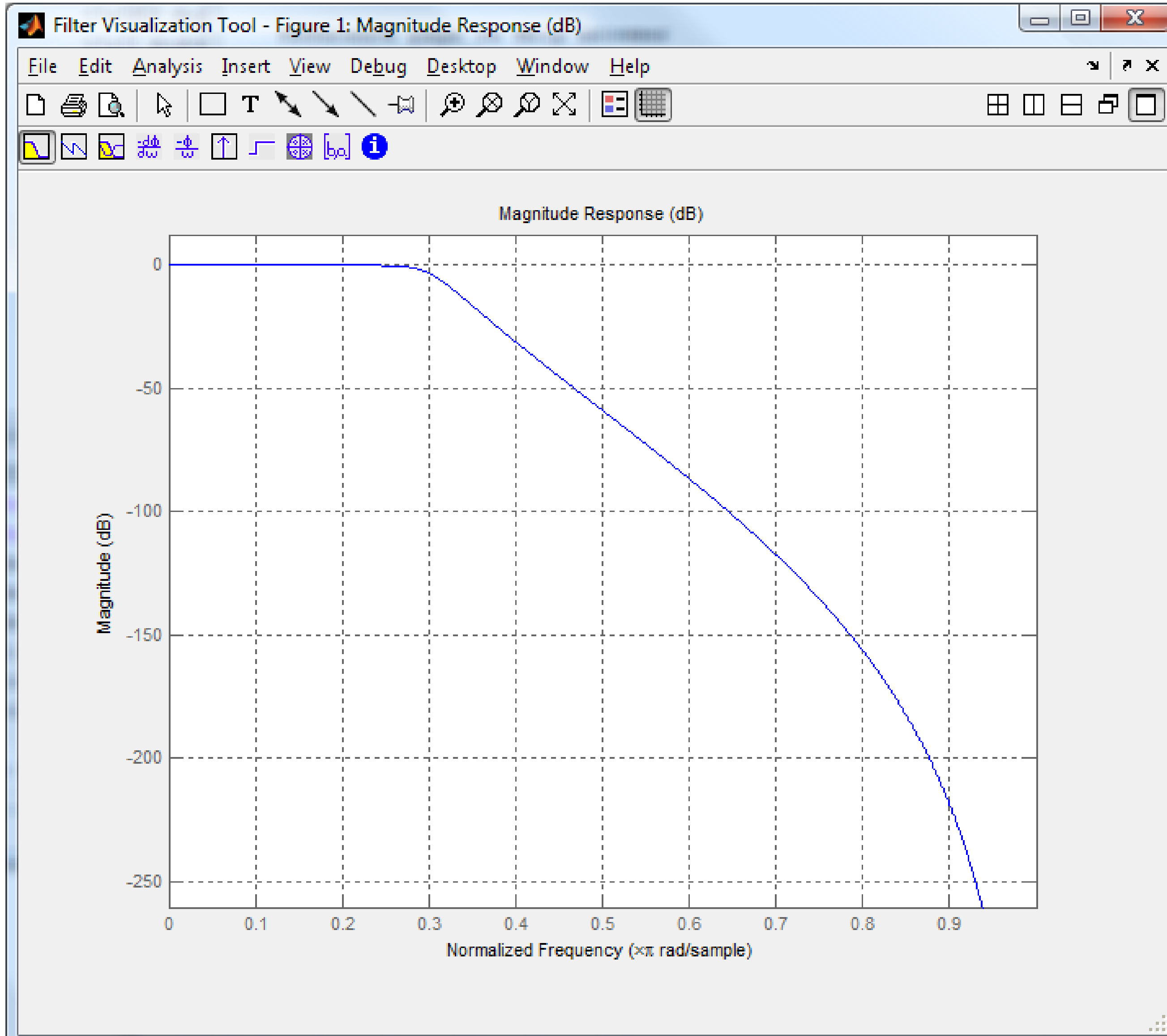
- Type **ver** to check
- Type **help** to locate help specific to Signal Processing Toolbox

freqz

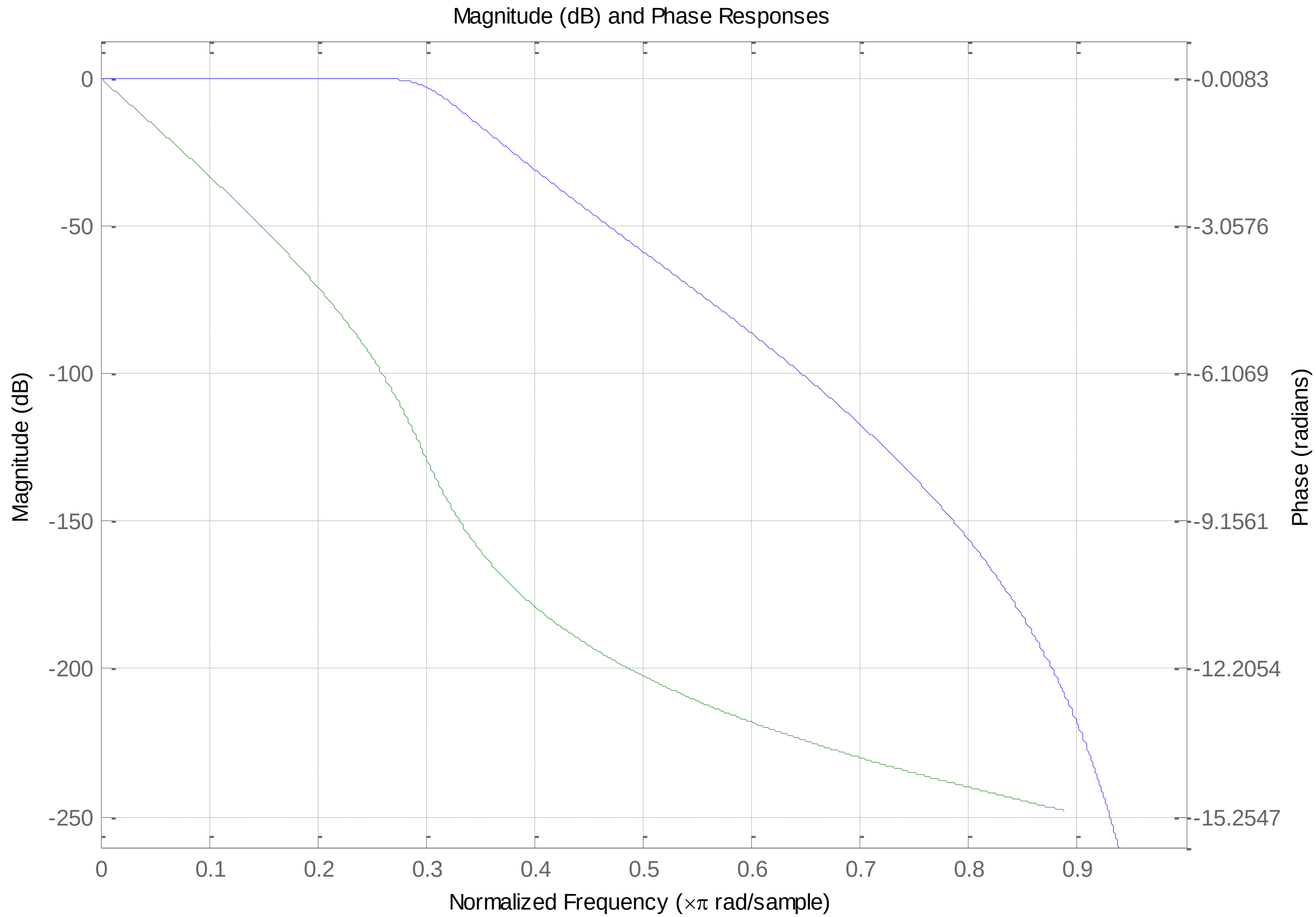


200
0
-200
-400
Magnitude (dB)
0
50
100
150
200
250
300
350
400
450
500
Frequency (Hz)
0
-500
-1000
Phase (degrees)
0
50
100
150
200
250
300
350
400
450
500
Frequency (Hz)

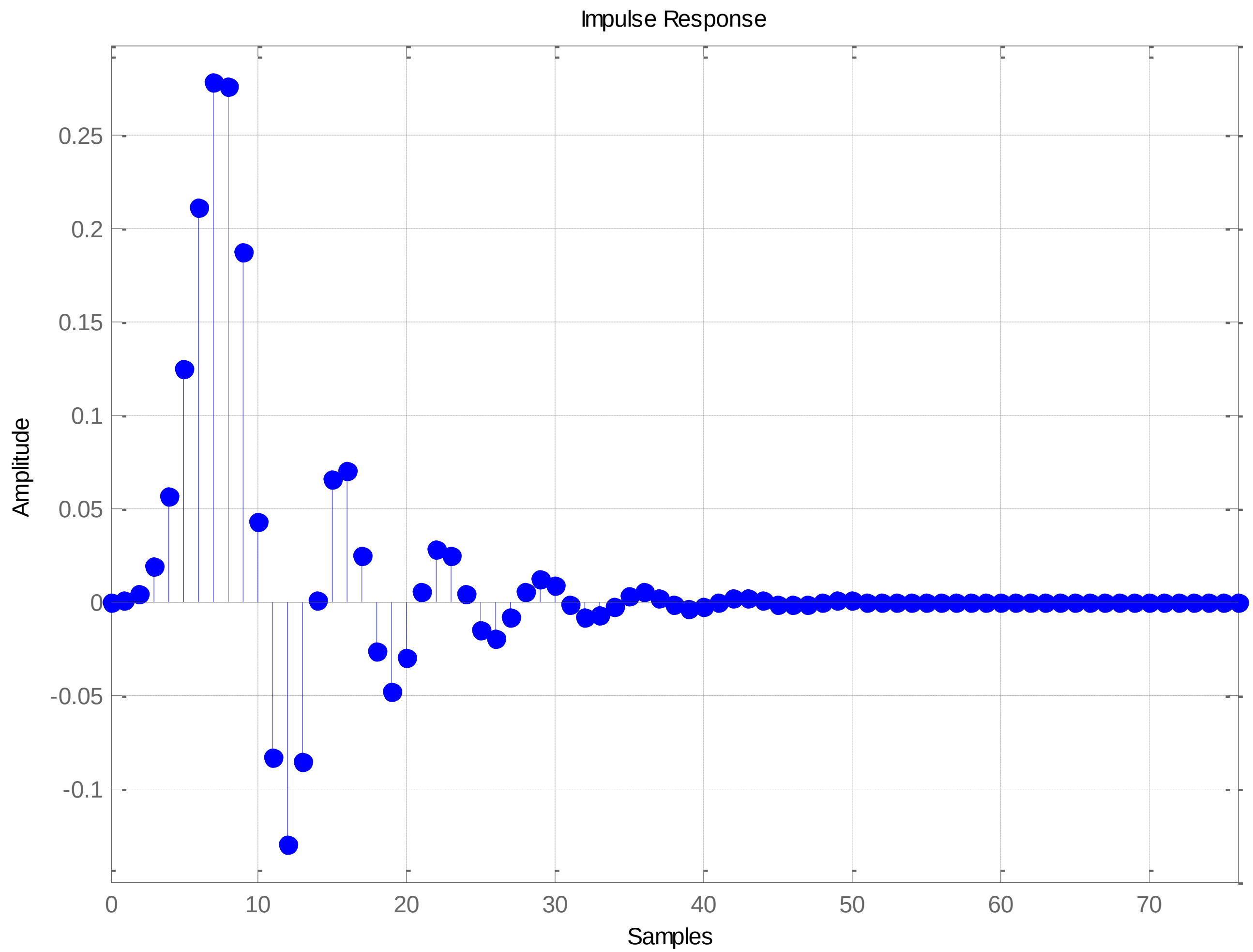
fvtool


Filter Visualization Tool - Figure 1: Magnitude Response (dB)
File Edit Analysis Insert View Debug Desktop Window Help
Magnitude Response (dB)
Magnitude (dB)
0
-50
-100
-150
-200
-250
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9
Normalized Frequency (×π rad/sample)

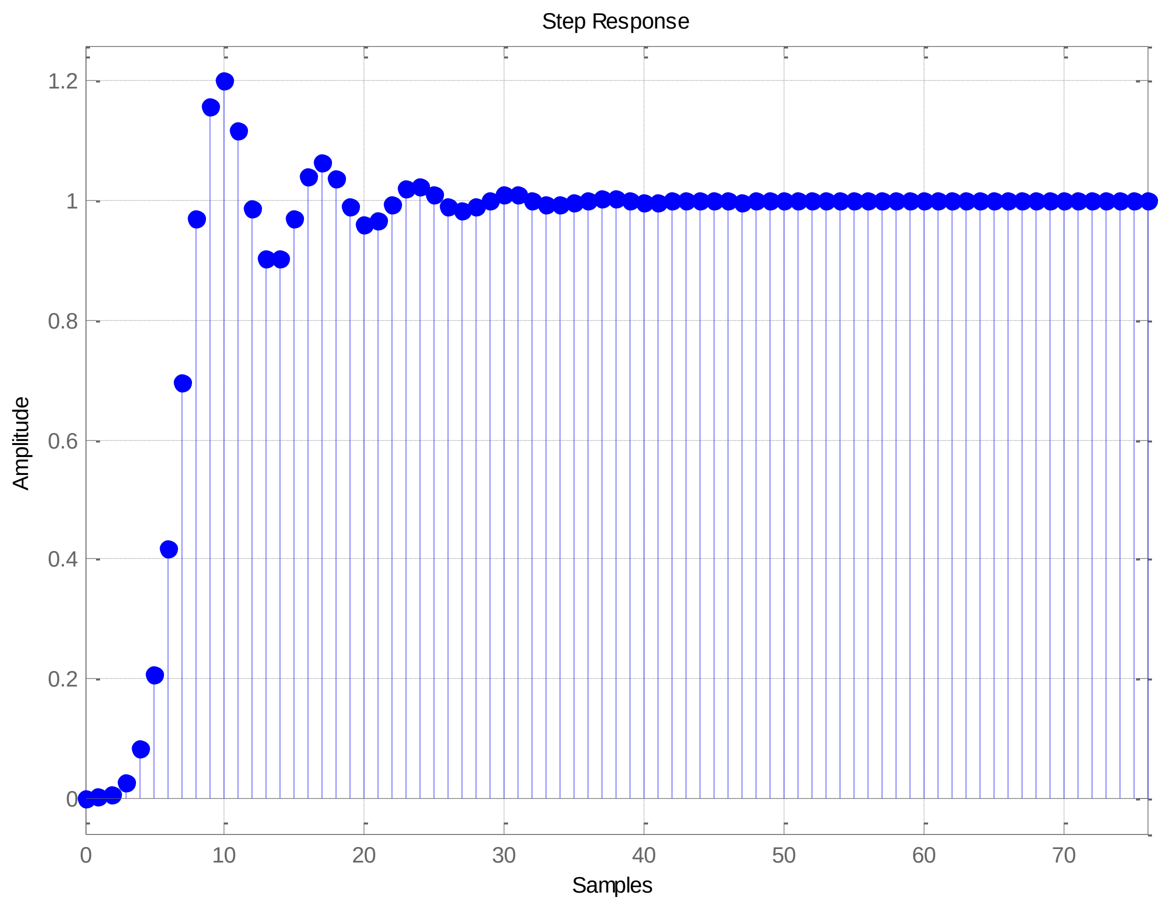
fvtool – magnitude and phase



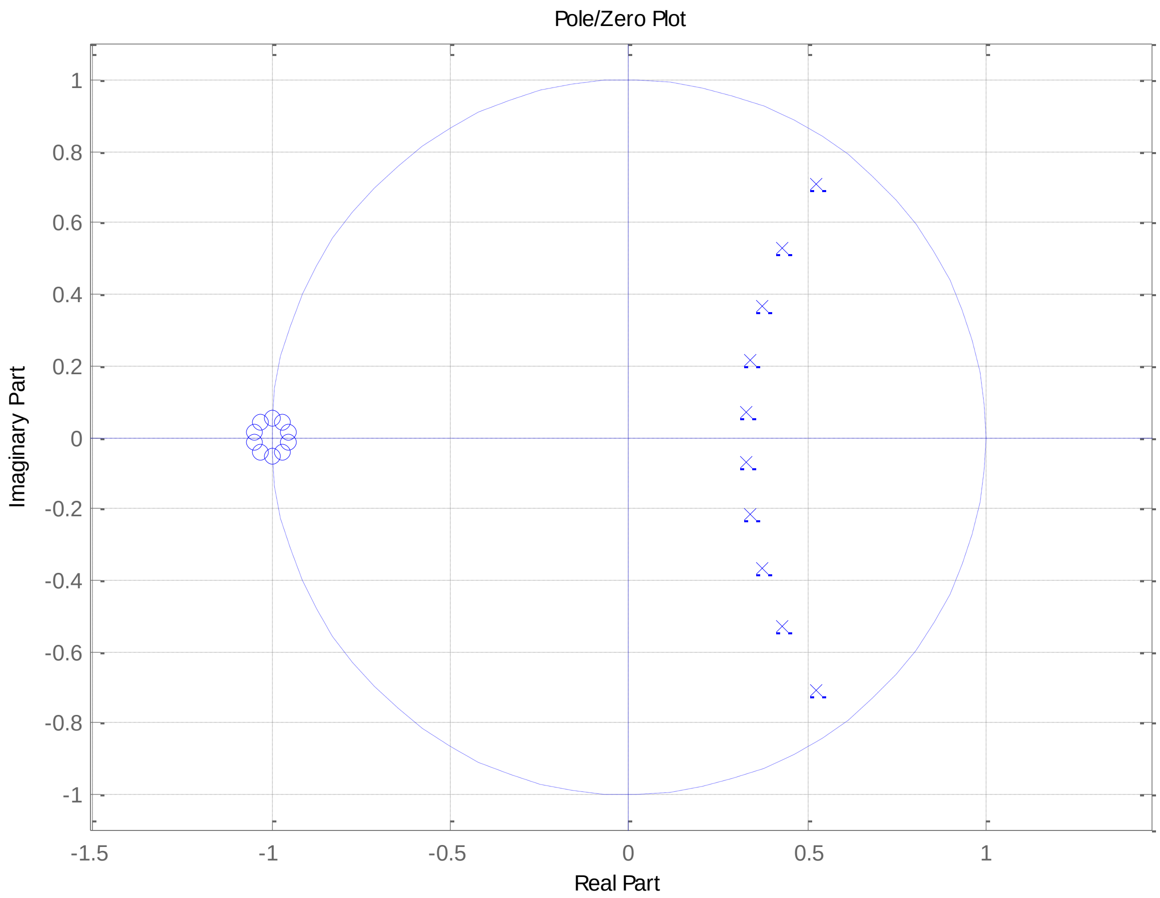
fvtool – impulse response



fvtool – step response

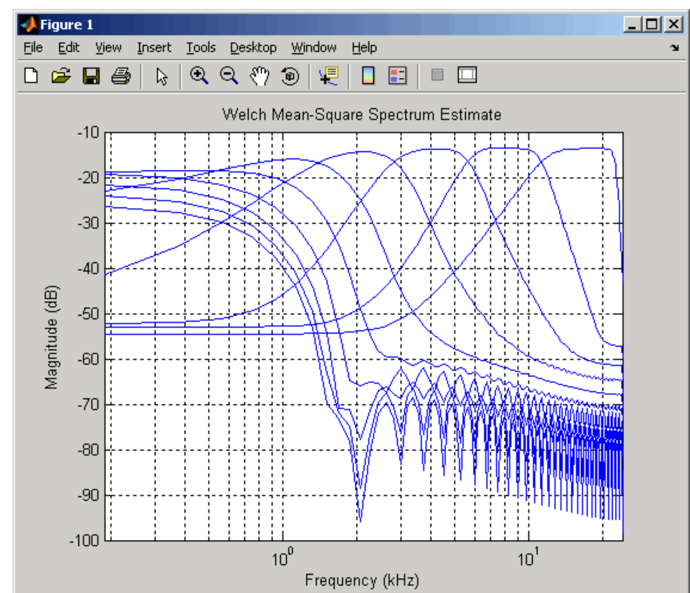


fvtool – pole/zero plot



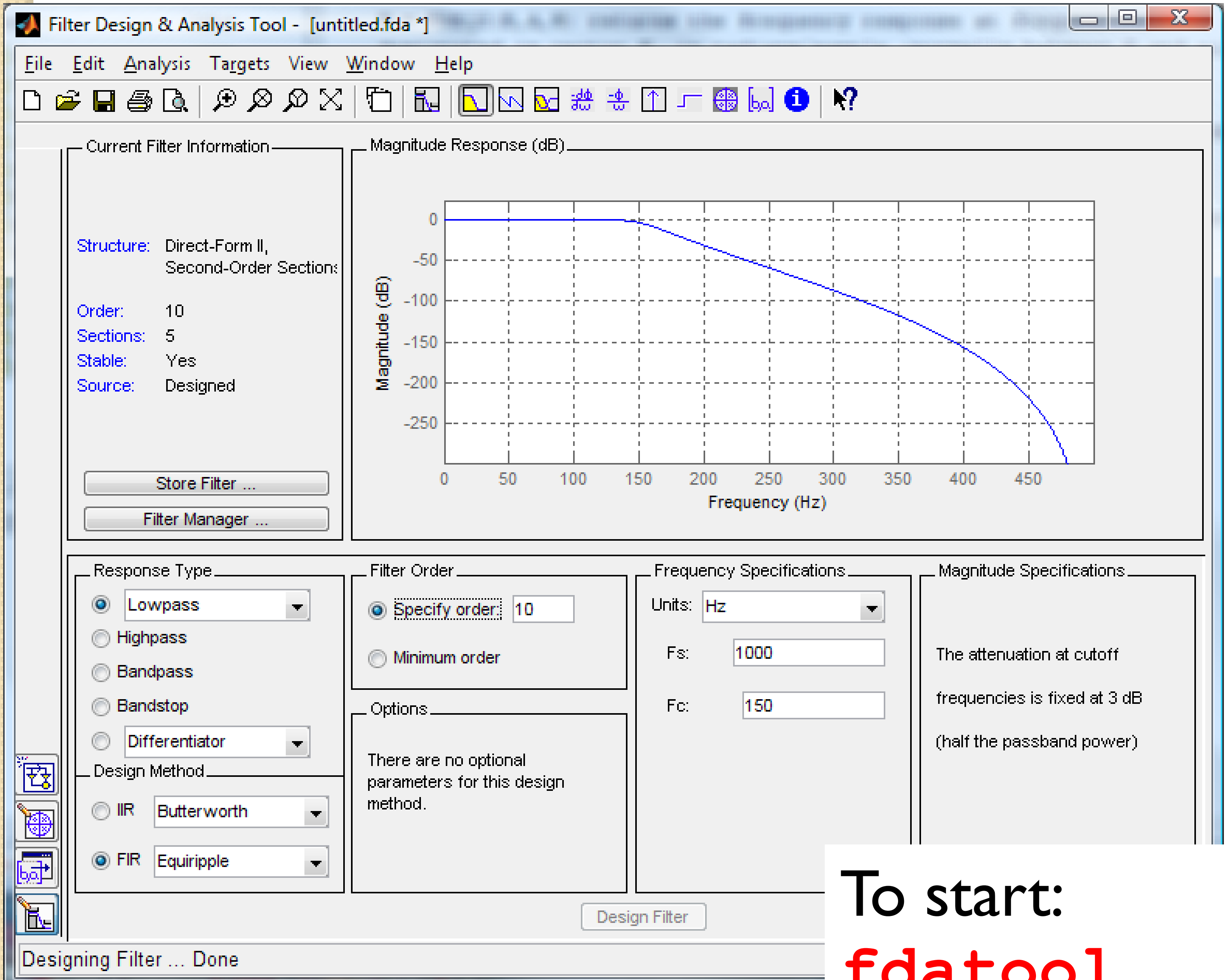
Signal Processing Toolbox

- FIR filter design
- Digital filter design
- Characterization/Analysis
- Implementation (convolution, etc.)
- Analog filters
- Waveform generators
- Some GUI tools



Fundamentals

- Represent signals as vectors
- Step is all 1s
- Impulse is a 1 followed by all 0s
- Several GUI tools are available:
    - sptool
    - fvtool
    - fdatool



To start:

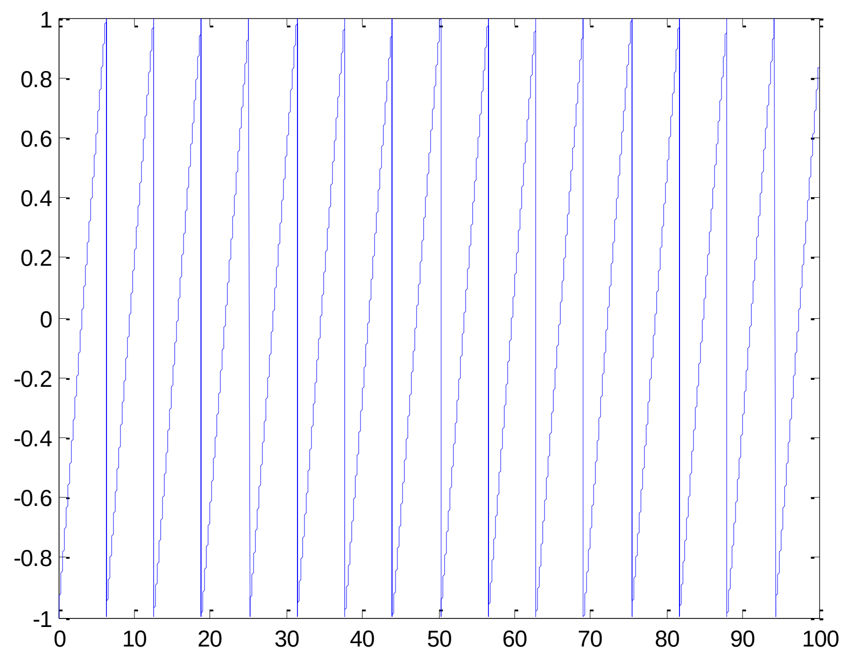
fdatool

Waveform Generators

- **sawtooth** - periodic sawtooth wave
- **square** – periodic square wave
- **tripuls** – single triangular pulse
- **rectpuls** - single rectangular pulse
- **gauspuls** – Gaussian-modulated sinusoidal pulse
- **sinc** – sin(x)/x
- **chirp** – linear, quadratic (convex or concave)
- **vco** – voltage controlled oscillator
- **pulstran** – pulse train (builds up train of any of the pulses above)
- For example: **pulstran(t,d,@rectpuls,w)** – d=delay times, w=pulse widths

Using Waveforms

- Sawtooth creates sawtooth wave with a width of 2*pi
- **t=0:0.001:100;**
- **y=sawtooth(t);**
- **plot(t,y)**



Spectral Analysis

- psd – power spectral density
- msspectrum – mean square
- pseudospectrum

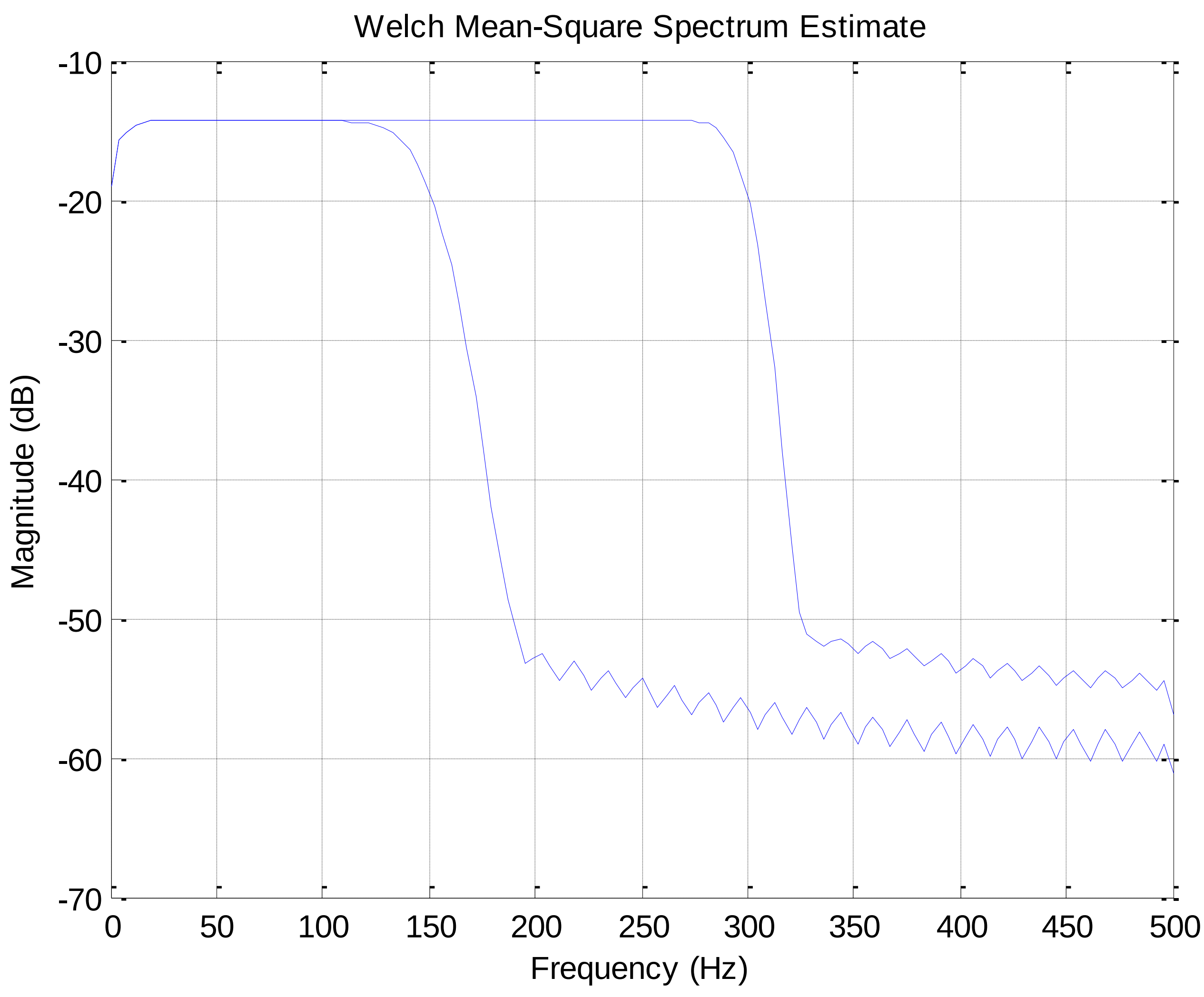
Create Spectral Analysis Object

- **h=spectrum.welch**
- Options include:
  - burg
  - cov-covariance
  - mcov-modified covariance
  - periodogram
  - welch
  - yulear – Yule-Walker autoregressive
- **mypower=msspectrum(h,y,'Fs',Fs)**
- **plot(mypower)**

The Script

```
h=spectrum.welch
mypower=msspectrum(h,y,'Fs',Fs)
plot(mypower)
mypowerfilt=msspectrum(h,yfilt,'Fs',Fs)
hold on
plot(mypowerfilt)
```

Result



Welch Mean-Square Spectrum Estimate
-10
-20
-30
-40
-50
-60
-70
Magnitude (dB)
0
50
100
150
200
250
300
350
400
450
500
Frequency (Hz)

Image Processing and cosine transforms

- You need the image processing toolbox
- I'll say a bit more about this toolbox later
- For now, let's look at the cosine transform
- This tool represents an image as a sum of sinusoids
- Much of the content of a figure is contained in just a small number of these sinusoids
- Hence, it is useful for image compression

Approach

- Read in image
- Take Discrete Cosine Transform
- Toss out higher order terms
- Compare result to original picture
- The built-in function **dct2** uses an FFT-like algorithm to compute transform

Script

```
RGB = imread('shuttle.jpg');
I = rgb2gray(RGB);
figure, imshow(I)
J = dct2(I);
J(abs(J) < 10) = 1e-8;
K = idct2(J);
figure, imshow(K,[0 255])
J = dct2(I);
J(abs(J) < 40) = 1e-8;
K = idct2(J);
figure, imshow(K,[0 255])
```

Statistics

- Transform matrix (J) originally has 288,960 elements (480x602)
- 181,697 have abs less than 10
- 274,221 have abs less than 40

First Compression (abs(J)<10)

First Compression (abs(J)<40)

Questions?